ON THE STABILITY OF PLANE-PARALLEL COUETTE FLOW

(OB USTOICHIVOSTI PLOSKOPARALLEL'NOGO TECHENIIA KUETTA)

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The problem of the stability of a plane-parallel flow of viscous incompressible fluid reduces, as is well known, to solution of the Orr-Sommerfeld equation [1]. The study of this equation presents great difficulty. Below we consider the very simple example when the flow velocity depends linearly on the transverse coordinate, i.e. the case of plane-parallel Couette flow. The mathematical problem reduces to determination of the sign of the imaginary part of the eigenvalues c of the boundary problem

$$(kz - c)(\varphi'' - \alpha^2\varphi) = \frac{1}{i\alpha R}(\varphi^{IV} - 2\alpha^2\varphi'' + \alpha^4\varphi)$$
$$\varphi(-1) = \varphi'(-1) = \varphi(1) = \varphi'(1) = 0 \qquad (1)$$

where k, α and R are certain real positive parameters. The number of parameters can indeed be reduced by setting $c_1 = c/k$ and $R_1 = Rk$, but it is more convenient to write the equation in the form (1). The parameters have the following physical significance: α is the wave length, c the phase velocity and R_1 is the Reynolds number. If the imaginary parts of all the eigenvalues c turn out to be negative, then the flow is stable.

There is a large number of papers in which an attempt is made to prove the stability of flow for all values of the parameters α and R_1. However, nobody has succeeded in proving this rigorously. The usual method of investigation is consideration of the asymptotic limiting cases, for example, large values of the Reynolds number and so on. For other values of the parameters a direct numerical calculation of the eigenvalues is made. In spite of the high state of development of asymptotic estimates attained at the present time (see, for example, [2]), such a method cannot in principle lead to a full and final solution of the problem, since it is not possible to carry out numerical calculations for the infinite range of the parameters, for which the asymptotic treatment is not suitable. (With regard to asymptotic investigations and numerical calculations, see also [3 and 4]). From the foregoing it follows that the construction of a rigorous and logically complete proof of stability of plane Couette flow remains, as before, highly desirable.

Below we prove a more limited proposition, namely that all purely imaginary eigenvalues c lie in the lower half-plane. (For small values of the Reynolds number all the eigenvalues are purely imaginary. With increasing R_1 they in turn leave the imaginary axis, first pairing off and then changing to pairs of points disposed symmetrically relative to the imaginary axis). We notice that in [5 and 4] the stability of purely imaginary eigenvalues is also established by means of a combination of asymptotic considerations and numerical calculations.

T h e o r e m . Purely imaginary eigenvalues of problem (1) satisfy the inequality $\operatorname{Im} c < -\alpha / R$.

By this we shall not only prove the stability of perturbations, but also give a certain estimate of their rate of damping.

Let us write $d = -ic + \alpha / R$, $\gamma = \alpha / k$, $a = \alpha R / k^2$. It will be necessary to prove that the real eigenvalues of d are negative. If we intriduce a new unknown function $u = \varphi'' - \alpha^2 \varphi$, then its governing equation will no longer be of the fourth, but of the second order

$$u'' - ia\xi u = 0 \qquad (\xi = kz - id)$$

(Airy's equation), but the boundary conditions will be replaced by certain integral conditions. It is not difficult to show that the eigenvalues of d are obtained as the roots of the secular equation

$$\Delta(k) \equiv \int\limits_{-id-k}^{-id+k} \int\limits_{-id-k}^{-id+k} \sinh \gamma (\xi - \xi_1)\, u_1(\xi)\, u_2(\xi_1)\, d\xi d\xi_1 = 0 \tag{2}$$

where u_1 and u_2 are any two linearly independent solutions of Airy's equations.

Let us expand the function $\Delta(k)$ as a series in powers of k, regarding γ and a as independent parameters. In an obvious way the function $\Delta(k)$ is an analytic function of the variable k. By means of the expansion of $\Delta(k)$ in powers of k, once it is obtained in explicit form, we shall show that Δ cannot vanish for positive values of the parameters γ, a, d and k, which proves the theorem. For the expansion in series it will be necessary to calculate successive derivatives of the function $\Delta(k)$ when $k = 0$. Let us write

$$W^{(m,n)}(\alpha, \beta) = u_1^{(m)}(\alpha)\, u_2^{(n)}(\beta) - u_1^{(n)}(\beta)\, u_2^{(m)}(\alpha)$$

Now let us differentiate

$$\frac{d\Delta(k)}{dk} = \int\limits_{-id-k}^{-id+k} [\sinh\gamma(-id + k - \xi)\, W^{(0,0)}(-id + k, \xi) - \sinh\gamma(-id - k - \xi)\, W^{(0,0)}(\xi, -id - k)]\, d\xi$$

In this expression k appears not only in the limits of integration, but also in the integrand. Therefore, in the next differentiation we shall obtain terms containing the integral and also non-integral terms – from the differentiation of the integral with respect to its limits of integration. In just the same way with each following differentiation there will remain some integral terms – from differentiation of the preceding integral term, and there will arise "new" non-integral terms – from differentiation of the same preceding integral term with respect to the limits of integration, and also supplementary non-integral terms – from differentiation of non-integral terms which had arisen earlier. By induction it can be proved that the integral terms and the "new" non-integral terms in the derivatives $d^p\Delta/dk^p$ are

$$\int\limits_{-id-k}^{-id+k} \Big\{\sinh\gamma(-id + k - \xi) \sum_{m=0}^{p-1} \binom{p-1}{m} W^{(m,\, p-1-m)}(-id + k, \xi) + \tag{3}$$
$$+ (-1)^p \sinh\gamma(-id - k - \xi) \sum_{m=0}^{p-1} \binom{p-1}{m} W^{(m,\, p-1-m)}(\xi, -id - k)\Big\}\, d\xi +$$
$$+ 2\,[1 + (-1)^p] \sinh 2\gamma k \sum_{m=0}^{p-2} \binom{p-2}{m} W^{(m,\, p-2-m)}(-id + k, -id - k)$$

We notice now that the last sum in this expression can be written as

$$i^{p-2}\partial^{p-2}W^{(0,0)}(-id+k,-id-k)\,/\,\partial d^{p-2}.$$

In order to write down now the complete derivative $d^p\Delta/dk^p$, we have to add to the term (3) the non-integral terms emerging in all the preceding differentiations, differentiated the appropriate number of times. We need the derivative when $k=0$.

If we set $k=0$, then the integral terms vanish, and we shall have

$$\frac{d^{2p+2}\Delta(0)}{dk^{2p+2}}=4\sum_{q=0}^{p}(-1)^q\frac{\partial^{2q}}{\partial d^{2q}}\left\{\frac{\partial^{2(p-q)}}{\partial k^{2(p-q)}}\left[\sinh 2\gamma k W^{(0,0)}(-id+k,-id-k)\right]\right\}_{k=0}$$

The derivatives of odd order are equal to zero. Let us re-write this formula as follows:

$$\frac{d^{2p+2}\Delta(0)}{dk^{2p+2}}=4\sum_{q=0}^{p-1}\sum_{r=0}^{p-q-1}(-1)^q\binom{2(p-q)}{2r+1}(2\gamma)^{2(p-q-r)-1}\times$$
$$\times\frac{\partial^{2q}}{\partial k^{2q}}\left[\frac{\partial^{2r+1}}{\partial k^{2r+1}}W^{(0,0)}(-id+k,-id-k)\right]_{k=0}\qquad(4)$$

Now let us calculate the expression in square brackets. For this we write

$$w_1=W^{(0,0)}(-id+k,-id-k),\qquad w_2=W^{(1,1)}(-id+k,-id-k)$$
$$w_{3,4}=W^{(1,0)}(-id+k,-id-k)\mp W^{(0,1)}(-id+k,-id-k)$$

Using Airy's equation, which u_1 and u_2 satisfy, it is not difficult to see that

$$\frac{\partial w_1}{\partial k}=w_3,\qquad \frac{\partial w_2}{\partial k}=-adw_3+iakw_4$$
$$\frac{\partial w_3}{\partial k}=2adw_1-2w_2,\qquad \frac{\partial w_4}{\partial k}=2iakw_1$$

Accordingly, the four functions satisfy a system of four equations. It can immediately be verified that these functions also satisfy the initial conditions

$$w_1=w_2=w_4=0,\qquad w_3=w,\qquad \text{for } k=0$$

Here the constant w can be taken as arbitrary, which corresponds to the arbitrariness in the choice of the two linearly independent solutions u_1 and u_2. We can assume that w_1, w_2 and w_4 are odd functions, whilst w_3 is even, and this is in any case clear immediately from formulas (5). We shall seek a solution of the system in the form of series in powers of k

$$w_1=\sum_{r=0}^{\infty}\frac{k^{2r+1}}{(2r+1)!}a_r\qquad\text{and so on.}$$

After substitution in the equations we obtain a recurrence formula for the coefficients

$$a_{r+3}=4ada_{r+2}+4a^2(2r+2)(2r+4)a_r\qquad(r\geqslant-2)$$

This difference equation determines a_r uniquely to within an arbitrary constant factor. A simple verification shows that this difference equation is satisfied by Expression

$$a_r=\sum_{s=0}^{r}{}'(4a)^{r-\frac{r-s}{3}}\frac{r!}{s!}3^{\frac{s-r}{3}}\left(\frac{r-s}{3}!\right)^{-1}$$

where the dash on the summation sign indicated that the sum is taken only over such values of the number s that $r-s$ is divisible by 3. Accordingly,

$$\left[\frac{\partial^{2r+1}}{\partial k^{2r+1}} W^{(0,0)}(-id+k, -id-k)\right]_{k=0} = \sum_{s=0}^{r}{}' (4a)^{r-\frac{r-s}{a}} \frac{r!}{s!} 3^{\frac{s-r}{3}} \left(\frac{r-s}{3}!\right)^{-1}$$

Formula (4) can now be written

$$\frac{d^{2p+2}\Delta(0)}{dk^{2p+2}} = 4\sum_{q=0}^{p-1}\sum_{r=0}^{p-q-1}(-1)^q\binom{2(p-q)}{2r+1}(2\gamma)^{2(p-q-r)-1} \times$$
$$\times \sum_{s=0}^{r}{}' (4a)^{r-\frac{r-s}{3}} \frac{r!d^{s-2q}}{(s-2q)!} 3^{\frac{s-r}{3}} \left(\frac{r-s}{3}!\right)^{-1}$$

It is more convenient to group the terms with the same powers of γ and d. To do this we write $s-2q=m$ and $p-q-r=n$ (it is easy to see that $p-n-m$ is divisible by 3, since $r-s$ is divisible by 3). As a result we have

$$\Delta = \sum_{p=1}^{\infty}\left(\sum_{m=0}^{p-1}\sum_{n=1}^{p-m}{}' a_{p,m,n} d^m \gamma^{2n-1}\right) k^{2p+2} \tag{6}$$

where only such values of n are taken that $p-n-m$ is divisible by 3, and

$$a_{p,m,n} = \frac{2^{2n+1}}{(2p+2)!}(4a)^{p-n-\nu}\sum_{q=0}(-1)^q\binom{2(p-q)}{2n-1}\frac{(p-n-q)!}{3^{\nu-q}(\nu-q)!\,m!}; \left(\nu = \frac{p-n-m}{3}\right)$$

The last expression is a series with alternating signs. It is elementary to prove that the terms of this series decrease monotonically in absolute magnitude, i.e. the sum is positive. Hence it follows that, for positive values of d, γ and a all the coefficients in the expansion of Δ in powers of k are positive, which also proves the theorem, since a power series with positive coefficients cannot vanish for positive values of the argument.

N o t e   o n   t h e   l i m i t i n g   c a s e $R \to \infty$ (absence of viscosity). The equation degenerates to $\varphi'' - \alpha^2\varphi = 0$, i.e. it has no solution satisfying the boundary conditions $\varphi(-1) = \varphi(1) = 0$. This means that the unsteady problem of stability cannot be reduced by separation of the variables to a problem in eigenvalues. Therefore, instead of the Orr-Sommerfeld equation it is necessary to study the problem with initial data for the equation containing the time variable. This equation is

$$\left(-\frac{i}{\alpha}\frac{\partial}{\partial t} + kz\right)\left(\frac{\partial^2}{\partial z^2} - \alpha^2\right)\varphi(z, t) = 0$$

The boundary conditions here are

$$\varphi(-1, t) = \varphi(1, t) = 0$$

and the initial conditions are $\varphi(z, 0) = \varphi_0(z)$

The solution is found immediately to be

$$\varphi(z, t) = -\frac{\sinh\alpha(z+1)}{\alpha\sinh 2\alpha}\int_z^1 e^{-i\alpha k z_1 t}\sinh\alpha(1-z_1)(\varphi_0'' - \alpha^2\varphi_0)\,dz_1 -$$
$$-\frac{\sinh\alpha(1-z)}{\alpha\sinh 2\alpha}\int_{-1}^{z} e^{-i\alpha k z_1 t}\sinh\alpha(z_1+1)(\varphi_0'' - \alpha^2\varphi_0)\,dz_1$$

From this formula it is evident that the solution as $t \to \infty$ is bounded, i.e. the flow is stable. A more detailed account of the stability of inviscid fluid is to be found in [6].

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